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Optimal design of a two-layered elastic strip subjected to transient loading

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Abstract

Analytical solutions play a very important role in the validation of numerical codes. However, exact analytical solutions involving optimal design of transiently loaded multilayered structures, are rare in the literature.

In this paper, we solve an optimal design problem involving wave propagation in a two-layered elastic strip subjected to transient loading. We obtain explicit formulas for the stress in each layer using the method of characteristics, and then use these results to identify the designs that provide the smallest stress amplitude. The derived analytical results are then successfully used to validate a previously developed, hybrid computational optimization software.

Keywords: Wave propagation; Goupillaud-type layered media; Structural optimization; Analytical solutions; Validation of numerical codes

1. Prior work

The study of wave propagation in layered media is important in a number of diverse disciplines such as seismology, electromagnetics, optics, acoustics, et cetera. An exhaustive review of the literature on the subject is beyond the scope of this paper, and the reader is referred to the classic treatise by Brekhovskikh (1960) for a review of the mathematical framework and physical phenomena related to wave propagation in elastic and electromagnetic media. The different approaches to the design of layered media for optimal reflection/transmission characteristics, can be broadly classified into either frequency domain or time domain methods. Some relevant work in this area is included below. Using frequency domain methods, Hager et al. (2000) derive relations needed to minimize or maximize reflection in elastic coatings subjected to steady-state, band-limited, time-harmonic acoustic waves. Hager and Rostamian (1987) also investigate conditions necessary to minimize acoustic reflections from a wall with a viscoelastic coating. Wesolowski

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(1995) studies a problem for a two-layer interface sandwiched between homogeneous elastic layers subjected to time-harmonic waves. Gusev (2001, 2002) provides extremal relations for the optimal design of homogeneous and inhomogeneous layered structures subjected to time-harmonic acoustic waves. Abgaryan and Lyubashevskii (1984) develop relations for active suppression of time-harmonic waves in a structure containing a nondirectional sensor and radiator. Krasil'nikov (1983) uses the method of "slowly varying amplitudes" to find optimal designs for a multilayered interference absorber. Bao and Bonnetier (2001) consider an optimal design problem of a periodic structure subjected to time-harmonic, transverse magnetically polarized waves and establish the existence of the optimal designs for the class of problems investigated.

In contrast to the body of literature on time-harmonic wave propagation, there is much less published work on the optimization of layered media subjected to transient loading. Nygren et al. (1999) study the problem of maximizing the efficiency of elastic energy transmission through a layer of elastic junctions for application to percussive drilling. Konstanty and Santosa (1995) pose a transient optimal design problem for minimally reflective coatings in the time-domain, and solve the problem numerically using a BFGS updated secant method; their analysis is limited to problems with small interlayer impedance contrast. A number of other authors consider problems related to the propagation of stress waves in discretely layered or inhomogeneous media; however, these authors do not address the problem of optimization. For instance, Ali (1999) uses transmission line theory to obtain recurrence relation solutions for acoustic waves in a layered medium. Tenenbaum and Zindeluk (1992) develop exact algebraic expressions for the reflected wave in a semi-infinite layered homogeneous or inhomogeneous medium. Lee et al. (1975), provide some general expressions for the stress jump in both functionally graded and discretely layered media, and provide error estimates when a medium with continuous property variation is replaced by a medium consisting of a series of discrete homogeneous layers. Chiu and Erdogan (1999), and Scheidler and Gazonas (2002), solve several transient wave-propagation boundary-value problems for free-fixed and free–free boundary conditions in power-law and quadratic inhomogeneous one-dimensional media, using Laplace transform methods.

Anfinsen (1967) treats a boundary-value problem similar to our own, and finds optimal material properties which maximize/minimize the amplitude of the first transmitted stress wave along an elastic strip. Anfinsen develops difference equations and solves them using z-transform methods. In addition, Anfinsen determines asymptotic similarity relationships applicable for optimal two-layered designs. We corroborate Anfinsen's findings which show that increasing the impedance ratio of the layers tends to minimize the maximum stress on a design. Interestingly, despite the work of Anfinsen, our approach and results for the stress and optimal design of a two-layered elastic strip, appear to be entirely new. Furthermore, we use our analytical results to validate the hybrid DYNA3D/GLO optimization software (Gazonas and Randers-Pehrson, 2001), previously used to find optimal designs to a class of impact and penetration problems using constitutive behaviors representative of lightweight multilayered armors. Finally, we mention that Bruck (2000) develops a time history profile for the stress waves and discovers a time delay benefit when using functionally graded materials. In our work, we discover a similar benefit when using two-layered optimal designs other than the homogeneous. Such designs provide a time delay benefit when the peak values of stress are reached.

2. Analytical solutions for stress wave propagation and optimal design

We consider one-dimensional wave propagation in an isotropic elastic strip. The strip is assumed to be of finite length L, made of two layers of arbitrary lengths L_1 and L_2, so that $L_1 + L_2 = L$. The layer interface is located at the fixed position $x = L_1$, where $0 < L_1 < L$. The left face of the strip is subjected to a stress loading p, while the right face is fixed (Fig. 1).

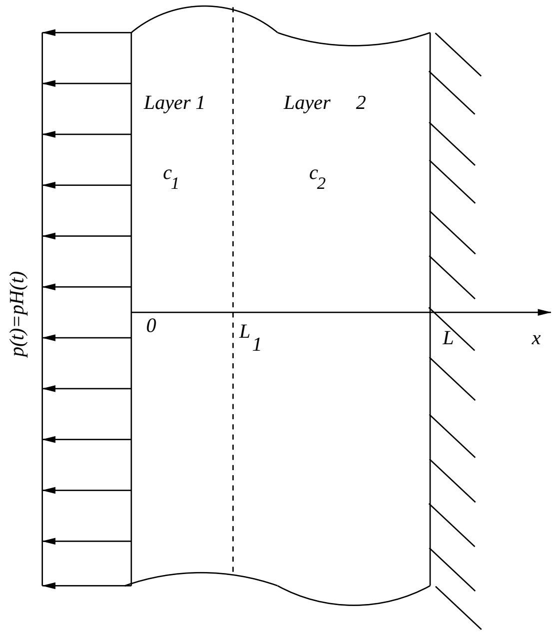

Fig. 1. Two-layered finite strip under transient load (physical coordinates).

The density and elastic modulus along the strip are denoted by the piece-wise constant functions $\rho(x)$ and $E(x)$, taking values ρ_1, ρ_2, and E_1, E_2, in each layer, respectively. In this work, we consider longitudinal elastic waves for a uniaxial strain state (Fig. 1), however our results are also valid for the case of uniaxial stress. Using the definitions of the wave speed $c = (E/\rho)^{1/2}$, and characteristic impedance $z = \rho c$, we relate to each layer the wave speed and characteristic impedance c_1, z_1 and c_2 and z_2, respectively. The uniaxial strain elastic modulus E, is related to the Young's modulus $\overline{E}$, through the expression $E = \overline{E}(1-v)/[(1+v)(1-2v)]$, where v is Poisson's ratio (Meyers, 1994).

Our design parameter α is chosen to represent the impedance ratio between layer 1 and layer 2: $\alpha = \frac{z_1}{z_2} > 0$. The transit time τ through the strip is given as

$$\tau = \frac{L_1}{c_1} + \frac{L_2}{c_2}.$$

Throughout this paper we assume a Goupillaud-type layered medium, which is a medium that ensures the same wave travel time through each layer, Claerbout (1976),

$$\frac{L_1}{c_1} = \frac{L_2}{c_2} = \frac{\tau}{2}. \tag{1}$$

In order to preserve the same wave travel time for both layers ($c_1/c_2 = L_1/L_2 = \mu$), the layer properties have to relate as: $\alpha = (\mu\rho_1)/\rho_2 = E_1/(\mu E_2)$. In the special case of two layers of equal length, $\mu = 1$ and the density and elastic modulus become proportional.

This problem can be easily converted to the case of two layers of equal length and wave speed of unity, by replacing the spatial variable x with the new variable $\xi = \int_0^x \frac{\mathrm{d}s}{c(s)}$, and using condition (1). Here, $c \equiv c(s)$ is the piece-wise constant wave speed function, taking values c_1 and c_2 in each layer, respectively. As a result, the wave equation becomes:

$$z\frac{\partial^2 u}{\partial t^2} = \frac{\partial\left(z\frac{\partial u}{\partial \xi}\right)}{\partial \xi}, \tag{2}$$

where $z = z(\xi)$ represents the characteristic impedance given by the piece-wise constant function:

$$z(\xi) = \begin{cases} z_1, & \text{where } 0 < \xi < \frac{\tau}{2}, \\ z_2, & \text{where } \frac{\tau}{2} < \xi < \tau. \end{cases} \tag{3}$$

The wave speed now becomes the same (unity) in each layer ($c = c_1 = c_2 = 1$), while the design parameter $\alpha = \frac{z_1}{z_2} > 0$, as stated before.

Let $\sigma_\alpha(\xi, t)$ represent the value of the stress at position ξ, time t, and design parameter α. Our goal is to find a design that provides the smallest stress amplitude during the wave propagation along the strip. We formulate our optimal design problem as

$$(\mathbf{P}) \quad \inf_{\alpha>0} \sup_{0\leqslant\xi\leqslant\tau,\, 0\leqslant t<+\infty} \sigma_\alpha(\xi, t),$$

subject to the initial/boundary-value problem,

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial \xi^2}, & \text{when } \xi \neq \frac{\tau}{2}, \\ \sigma(0,t) = z\frac{\partial u}{\partial \xi}(0,t) = pH(t), & u(\tau, t) = 0, \\ u(\xi, 0) = \frac{\partial u}{\partial t}(\xi, 0) = 0. \end{cases} \tag{4}$$

Here, $u(x, t)$ represents the displacement at (x, t), and $H(t)$ represents the Heaviside function. As demonstrated later in the paper, solving problem (**P**), becomes equivalent to solving the physical problem with layers of unequal lengths, because the only essential condition that influences the stress wave propagation along the strip, is the equal travel time through each layer. Therefore, from now on, any conclusions made for the case of two layers of equal length (Fig. 2), will apply to the general physical case of layers of unequal lengths (Fig. 1).

For the two-layered strip shown in Fig. 2, the propagation of the stress wave from layers 1 to 2, can be expressed by the following relations,

$$\sigma^- = \sigma^+ + [\sigma], \quad [\sigma]_\mathrm{T} = \frac{2}{1+\alpha}[\sigma]_\mathrm{I}, \quad [\sigma]_\mathrm{R} = \frac{1-\alpha}{1+\alpha}[\sigma]_\mathrm{I}. \tag{5}$$

Here, σ^- and σ^+ represent the stress values behind and ahead of the discontinuity, while $[\sigma]$ represents the stress jump. The subscripts refer to I (incident), T (transmitted), and R (reflected) wave. The above relations, see Meyers (1994), involve the continuity conditions at the layer interface, and will be our main reference in calculating the stress values throughout this paper.

In the special case of a homogeneous design, when $\alpha = 1$, combining (5) with the method of characteristics, one can easily derive the time history profile for the stress. In the region ahead and behind the stress discontinuity propagating along the strip, the stress takes the following values, respectively,

$$\begin{cases} 0 \text{ and } p, & \text{during the time interval } (0, \tau), \\ p \text{ and } 2p, & \text{during the time intervals } ((2k-1)\tau, 2k\tau) \text{ for } k = 1, 2, \ldots, \infty, \\ 2p \text{ and } p, & \text{during the time intervals } (2k\tau, (2k+1)\tau). \end{cases} \tag{6}$$

This implies that,

$$\max_{0<\xi<\tau,\, 0<t<+\infty} \sigma_1(\xi, t) = 2p, \tag{7}$$

where according to our notation, $\sigma_1(\xi, t)$ represents the stress at position ξ, time t, and design parameter $\alpha = 1$. From here, an upper bound for the optimization problem (**P**) follows,

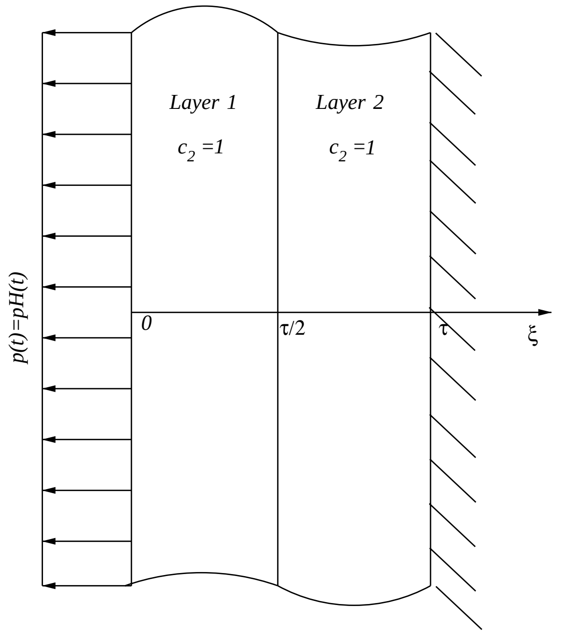

Fig. 2. Two-layered finite strip under transient load (transformed coordinates).

$$\inf_{\alpha>0} \sup_{0\leqslant\xi\leqslant\tau,\,0\leqslant t<+\infty} \sigma_\alpha(\xi,t) \leqslant \max_{0\leqslant\xi\leqslant\tau,\,0\leqslant t<+\infty} \sigma_1(\xi,t) = 2p. \tag{8}$$

Based on the results given later in this section, we prove that the reverse inequality in (8) holds, and conclude that the homogeneous design is an optimal design for problem (**P**). We also notice that no design with design parameter $\alpha < 1$, can be optimal, since for such designs the stress amplitude always exceeds the value $2p$. Indeed, following the first transmitted wave and using (5), it appears that in the region behind the stress discontinuity, the stress takes values,

$$\sigma(\xi,t) = \frac{2}{1+\alpha}p > p, \quad \text{where } \frac{\tau}{2} < t < \tau \text{ and } \xi = t.$$

As the first transmitted wave reflects at the fixed boundary located at $\xi = \tau$, the stress doubles and reaches a value greater than $2p$. Summarizing the above observations we have,

$$\sup_{0\leqslant\xi\leqslant\tau,\,0\leqslant t<+\infty} \sigma_\alpha(\xi,t) \geqslant 2p = \max_{0\leqslant\xi\leqslant\tau,\,0\leqslant t<+\infty} \sigma_1(\xi,t), \quad \text{where } 0 < \alpha \leqslant 1. \tag{9}$$

In the rest of this paper, we investigate and obtain optimality results for two-layered designs with $\alpha > 1$. In Fig. 3a and b, are given the Lagrangian diagrams for the case of equal layer length under consideration, as well as for the general physical case of two layers of unequal lengths but equal travel time.

For a given $\alpha > 1$, after applying (5) and the method of characteristics, we find the following recurrence relation among the stress values $T_{\alpha,k}$,

$$\begin{cases} T_{\alpha,k} = \beta(\alpha)\cdot(T_{\alpha,k-1} - T_{\alpha,k-2}) + T_{\alpha,k-3}, & k = 4,5,\ldots,\infty, \\ T_{\alpha,1} = \dfrac{2}{(\alpha+1)}p, \quad T_{\alpha,2} = \dfrac{8\alpha p}{(\alpha+1)^2}, \quad T_{\alpha,3} = \dfrac{2(3\alpha-1)^2 p}{(\alpha+1)^3}, \end{cases} \tag{10}$$

where $\beta(\alpha) = \frac{3\alpha-1}{\alpha+1}$. The recurrence relation given in (10) is linear, homogeneous and with constant coefficients. Its corresponding characteristic equation,

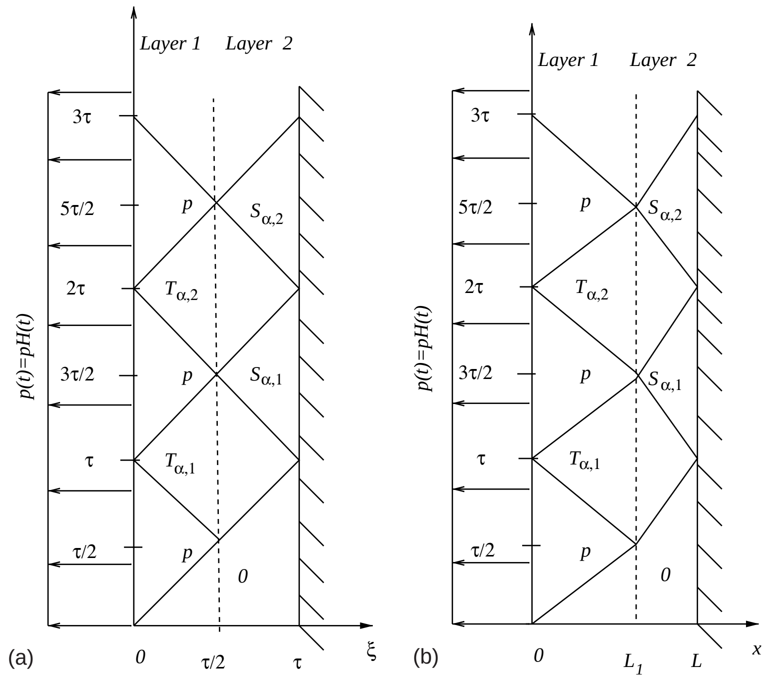

Fig. 3. Lagrangian diagram of stress waves: (a) equal layer length, (b) unequal layer length.

$$T^k - \beta(\alpha) \cdot T^{k-1} + \beta(\alpha) \cdot T^{k-2} - T^{k-3} = 0,$$

has three distinct roots,

$$y_1 = 1, \quad y_2 = \frac{(\alpha - 1) + 2\sqrt{\alpha}\mathscr{I}}{\alpha + 1}, \quad \text{and} \quad y_3 = \frac{(\alpha - 1) - 2\sqrt{\alpha}\mathscr{I}}{\alpha + 1}, \tag{11}$$

where $\mathscr{I} = \sqrt{-1}$. The general solution is

$$T_{\alpha,k} = A_1 y_1^k + A_2 y_2^k + A_3 y_3^k \tag{12}$$

with coefficients A_1, A_2, A_3 determined from the boundary conditions given in (10),

$$A_1 = p, \quad A_2 = -\frac{p}{2}, \quad A_3 = -\frac{p}{2}. \tag{13}$$

Substituting (11) and (13) into (12), we obtain,

$$T_{\alpha,k} = p - \frac{p}{2}\left[\frac{(\alpha - 1) + 2\sqrt{\alpha}\mathscr{I}}{\alpha + 1}\right]^k - \frac{p}{2}\left[\frac{(\alpha - 1) - 2\sqrt{\alpha}\mathscr{I}}{\alpha + 1}\right]^k. \tag{14}$$

Further simplifications of (14), lead to the following expression for the term $T_{\alpha,k}$,

$$T_{\alpha,k} = p[1 - \cos(k\varphi(\alpha))], \tag{15}$$

where $\varphi(\alpha) = \arctan\frac{2\sqrt{\alpha}}{\alpha-1}$, and $\alpha > 1$. In fact, the following relations are equivalent,

$$\varphi = \arctan\frac{2\sqrt{\alpha}}{\alpha - 1}, \quad \text{where } \alpha > 1 \tag{16}$$

and,

$$\alpha = \frac{1+\cos\varphi}{1-\cos\varphi}, \quad \text{where } 0 < \varphi < \frac{\pi}{2}. \tag{17}$$

From (15), one can derive the following bounds for the stress amplitude in layer 1,

$$0 \leqslant \sup_{0<\xi<\frac{\tau}{2},\ 0<t<+\infty} \sigma_\alpha(\xi,t) \leqslant 2p, \quad \text{where } \alpha \geqslant 1. \tag{18}$$

These results indicate that for all designs with $\alpha \geqslant 1$, the stress amplitude in layer 1 never exceeds the value $2p$.

Similarly, using the continuity conditions at the layer interface and the method of characteristics, we derive that the new stress terms $S_{\alpha,k}$ satisfy the following relation,

$$\begin{cases} S_{\alpha,k} = -S_{\alpha,k-1} + 2T_{\alpha,k}, & k = 1,2,\ldots,\infty, \\ S_{\alpha,1} = 2T_{\alpha,1} = \frac{4}{(\alpha+1)}p, \end{cases} \tag{19}$$

or equivalently,

$$\begin{cases} S_{\alpha,k} = 2(-1)^k \sum_{i=1}^{k} (-1)^i T_{\alpha,i}, & k = 1,2,\ldots,\infty, \\ S_{\alpha,1} = 2T_{\alpha,1} = \frac{4}{(\alpha+1)}p. \end{cases} \tag{20}$$

Substituting the expression for $T_{\alpha,k}$ in (20), we further obtain,

$$S_{\alpha,k} = 2(-1)^k p \sum_{j=1}^{k} (-1)^j - 2(-1)^k p \sum_{j=1}^{k} (-1)^j \cos(j\varphi(\alpha)), \tag{21}$$

where as before $\varphi(\alpha) = \arctan(\frac{2\sqrt{\alpha}}{\alpha-1})$, and $\alpha > 1$. The calculation of each partial sum and further simplifications imply that the term $S_{\alpha,k}$ is given by the formula,

$$S_{\alpha,k} = p\left[1 - \frac{\cos\left((2k+1)\frac{\varphi(\alpha)}{2}\right)}{\cos\frac{\varphi(\alpha)}{2}}\right]. \tag{22}$$

Due to the above expression for the terms $S_{\alpha,k}$, the stress amplitude in layer 2 will either reach or exceed the value $2p$, for any given design with $\alpha > 1$. A more detailed discussion is provided when the set K_α is introduced in (26). As a result we have,

$$\sup_{\frac{\tau}{2}<\xi<\tau,\ 0<t<+\infty} \sigma_\alpha(\xi,t) \geqslant 2p, \quad \text{where } \alpha \geqslant 1. \tag{23}$$

Combining (18) and (23) we have that,

$$\sup_{0\leqslant\xi\leqslant\tau,\ 0<t<+\infty} \sigma_\alpha(\xi,t) \geqslant 2p, \quad \text{where } \alpha \geqslant 1. \tag{24}$$

Finally, combining (8), (9) and (24), we conclude that,

$$\inf_{\alpha>0} \sup_{0\leqslant\xi\leqslant\tau,\ 0\leqslant t<+\infty} \sigma_\alpha(\xi,t) = \max_{0\leqslant\xi\leqslant\tau,\ 0\leqslant t<+\infty} \sigma_1(\xi,t) = 2p. \tag{25}$$

This clearly indicates that the homogeneous design with $\alpha = 1$, is an optimal design. Generally, a design with $\alpha > 1$ is optimal iff the stress values given by $S_{\alpha,k}$ for $k = 1,2,\ldots,\infty$, do not exceed $2p$.

Indeed, for any given $\alpha > 1$, we identify the set K_α as the set of all natural numbers $k = k(\alpha)$ such that $S_{\alpha,k} \geqslant 2p$. Based on the expression of the terms $S_{\alpha,k}$ given in (22), the condition $S_{\alpha,k} \geqslant 2p$, becomes equivalent to $\left[\frac{\cos((2k+1)\frac{\varphi(\alpha)}{2})}{\cos\frac{\varphi(\alpha)}{2}} \leqslant -1\right]$, and furthermore to $\left[(2j+1)\pi - \frac{\varphi(\alpha)}{2} \leqslant (2k+1)\frac{\varphi(\alpha)}{2} \leqslant (2j+1)\pi + \frac{\varphi(\alpha)}{2}\right]$, where $j = 0,1,2,\ldots,\infty$. As a result, we can describe the set K_α as

$$K_\alpha = \left\{ k : 2k+1 \in \left[(2j+1)\frac{2\pi}{\varphi(\alpha)} - 1, (2j+1)\frac{2\pi}{\varphi(\alpha)} + 1 \right],\ j = 0,1,2,\ldots,\infty \right\}. \quad (26)$$

For any given $\alpha > 1$, K_α is countably infinite because there is always at least one odd number of the form $2k+1$, in an interval of two-unit length such as $[(2j+1)\frac{2\pi}{\varphi(\alpha)} - 1, (2j+1)\frac{2\pi}{\varphi(\alpha)} + 1]$ for $j = 0,1,2,\ldots,\infty$. This proves that the terms $S_{\alpha,k}$, and therefore the stress amplitude, will reach or exceed the value $2p$ for some value of k, for every $\alpha > 1$. From here it follows that a design is optimal iff $S_{\alpha,k} = 2p$ for all $k \in K(\alpha)$. This is equivalent to the fact that $[2k+1 = (2j+1)\frac{2\pi}{\varphi(\alpha)} \pm 1]$ for $j = 0,1,2,\ldots,\infty$, which combined with (16), (17) and (25), leads to the following necessary and sufficient condition for the optimal values of the design parameter α,

$$\alpha_n = \frac{1+\cos(\varphi_n)}{1-\cos(\varphi_n)}, \quad \text{where } 0 < \varphi_n = \frac{\pi}{n} \leqslant \frac{\pi}{2} \text{ and } n = 2,3,\ldots,\infty. \quad (27)$$

Substituting these optimal values into the stress formulas $T_{\alpha,k}$ and $S_{\alpha,k}$, we find that these periodic functions have a time-period of $2n$, where the stress reaches its first peak of $2p$ at the nth time interval. These results, applied to the time history profiles involving the analytical solutions (15) and (22), match with the graphical output given in Figs. 4a and b and 5a and b, included in the next page. The normalized values for stress and time used in Figs. 4a and b and 5a and b, are calculated as, $\sigma^* = \frac{\sigma}{p}$, and $\mathscr{T} = \frac{t}{\tau}$.

Here p, τ are the initially given constants representing the stress loading, and the transit time through the elastic strip. We also notice that from (5), the stress jump at the optimal designs with parameter α_n takes values $[\sigma]_{\mathrm{T}} = (1-\cos(\frac{\pi}{n}))[\sigma]_{\mathrm{I}}$, and $[\sigma]_{\mathrm{R}} = -\cos(\frac{\pi}{n})[\sigma]_{\mathrm{I}}$. In general, for the designs with parameter,

$$\alpha_{j,n} = \frac{1+\cos\varphi_{j,n}}{1-\cos\varphi_{j,n}}, \quad \text{where } 0 < \varphi_{j,n} = \frac{(2j+1)\pi}{n} < \frac{\pi}{2}, \quad (28)$$

there are time intervals when the stress takes the value $2p$. Here j and n are natural numbers, chosen to satisfy the bounds for $\varphi_{j,n}$ given in (28).

The following examples, provide a means to better understand the structure of the K_α set and the conclusions made in this section.

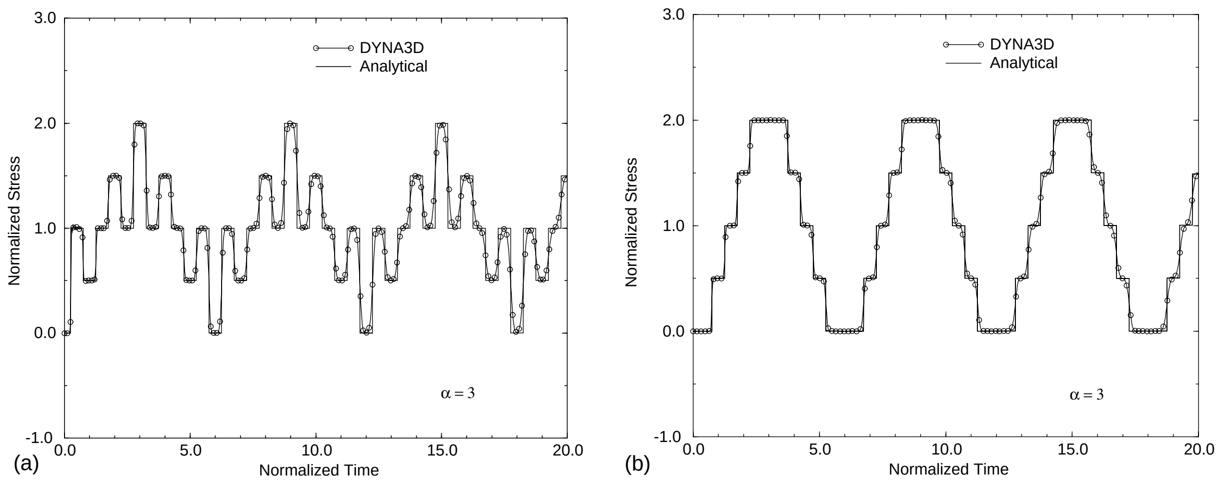

Fig. 4. Optimal stress history when $\tau = 1$ and $\alpha = 3$: (a) layer 1: $\xi = 0.25$, (b) layer 2: $\xi = 0.75$.

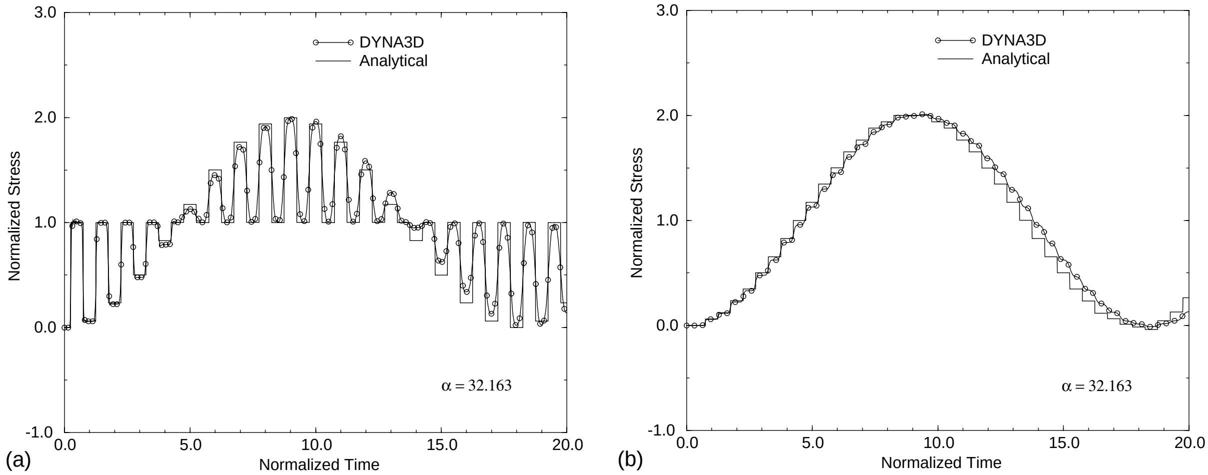

Fig. 5. Optimal stress history when $\tau = 1$ and $\alpha = 32.163$: (a) layer 1: $\xi = 0.25$, (b) layer 2: $\xi = 0.75$.

Example 1. $\alpha = \frac{1+\cos(\pi/3)}{1-\cos(\pi/3)} = 3$. According to (27), such a design obtained for $n = 3$, is optimal. In this case $\varphi(\alpha) = \frac{\pi}{3}$ and (26) implies that,

$$\begin{aligned} K_\alpha &= \{k : 2k+1 \in [6(2j+1)-1, 6(2j+1)+1],\ j = 0,1,2,\ldots,\infty\} \\ &= \{6j+2, 6j+3,\ j = 0,1,2,\ldots,\infty\}. \end{aligned} \tag{29}$$

Based on the structure of K_α, the stress amplitude reaches its peaks when $S_{\alpha,6j+2} = 2p$, and $S_{\alpha,6j+3} = 2p$. As predicted for $n = 3$, the first peak is reached at the 3rd time interval with 6 units time-period. This can be verified by the graphical output given in Fig. 4b.

Example 2. $\alpha = \frac{1+\cos(5\pi/11)}{1-\cos(5\pi/11)}$. Here $\varphi(\alpha) = \frac{5}{11}\pi$ and from (26) we obtain,

$$K_\alpha = \left\{k : 2k+1 \in \left[\frac{22(2j+1)}{5} - 1, \frac{22(2j+1)}{5} + 1\right],\ j = 0,1,2,\ldots,\infty\right\}. \tag{30}$$

Although the design parameter α is not optimal, it satisfies the condition (28) for $j = 2$ and $n = 11$. One can easily check that although $S_2 > 2p$ and $S_6 > 2p$, there are indices such as $\{10, 11\} \subset K_\alpha$ for which $S_{10} = 2p$ and $S_{11} = 2p$, as predicted.

Example 3. $\alpha = \frac{1+\cos(\pi/\sqrt{6})}{1-\cos(\pi/\sqrt{6})}$. In this case we have that $\varphi(\alpha) = \pi/\sqrt{6}$ and from (26) we obtain,

$$K_\alpha = \{k : 2k+1 \in [(4j+2)\sqrt{6} - 1, (4j+2)\sqrt{6} + 1],\ j = 0,1,2,\ldots,\infty\}. \tag{31}$$

In this case, none of the conditions (27) or (28) is satisfied. This means that the design is not optimal and that all the terms $S_{\alpha,k}$ for $k \in K_\alpha$, will exceed the value $2p$ without ever reaching it, i.e. we expect that $S_{\alpha,k} > 2p$, $\forall k \in K_\alpha$.

In conclusion, in this work, we solve the problem of minimizing the stress amplitude for a two-layered elastic strip, with layers of equal wave travel time, and subjected to transient loading. Our results provide explicit formulas for the stress propagation and optimal designs in one-dimension, and apply for a two-layered elastic strip of unequal layer lengths.

3. Numerical results using DYNA3D/GLO optimization software

Since exact analytical solutions to optimal design problems involving transiently loaded multilayered structures are rare in the literature, analysts have relied on hybrid computational methods which link formal nonlinear parameter estimation algorithms with computational finite element methods for designing structures subjected to transient loadings. The results of the previous section provide useful benchmarks which can be used to validate these hybrid optimization design tools. Such a design tool was recently developed by Gazonas and Randers-Pehrson (2001), by coupling the global-local optimization software package known as GLO (Murphy, 1999) with the three-dimensional transient structural dynamics finite element code known as DYNA3D (Hallquist and Whirley, 1989). The local optimization algorithm within GLO is based upon the nonlinear optimization program NLQPEB that was developed by Baker (1992) for optimization of shaped charge jets.

The optimization software was tested to see if it could find any of the optimal points given by the optimality condition (27). The boundary value problem illustrated in Fig. 2 was modeled using DYNA3D with 60 hexahedral finite elements through the thickness. The optimization strategy is based upon a global-local approach since the optimal points in layer 2 contain an infinite number of minima, four of which are illustrated in Fig. 6. The plots in Fig. 6 were determined using our analytical expressions for stress, $S_{\alpha,k}$ and $T_{\alpha,k}$, through parametric variation of α over the range 0–10, and k to a time range out to 50 units.

The global optimization uses a discrete scheme which simply subdivides the design parameter α into uniformly spaced regions from which the local scheme can begin searching for extrema. The local scheme seeks to minimize the maximum stress in layer 2 using a BFGS variable-metric, sequential-quadric-programming algorithm (Dennis and Schnabel, 1983), with a modified Powell merit function (Powell, 1977). Since the number of optimal designs is infinite in nature, the search was limited, rather arbitrarily, to a region involving the first eight optimal design points. Table 1 shows that the hybrid DYNA3D/GLO optimization software successfully found all of the first eight optimal design points to a reasonable degree of accuracy. This result both validates the DYNA3D/GLO optimization software and corroborates the optimality condition obtained earlier.

An example of optimal layer design for a two-layered elastic strip of equal lengths, is a design made of tungsten and lead ($\alpha = 3$) or a design made of tungsten and nylon ($\alpha = 32.163$). The proportionality between the density and elastic modulus in both layers, provide a constant wave speed and therefore equal travel

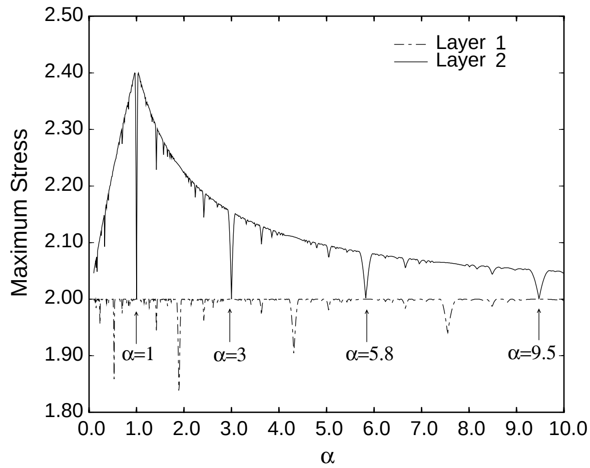

Fig. 6. Maximum stress vs. α in a two-layered finite strip showing the first four optimal α points given by optimality condition (27).

Table 1
Comparison of analytical and DYNA3D/GLO derived α values

n	2	3	4	5	6	7	8	9
α (Eq. (27))	1	3	5.828	9.472	13.983	19.196	25.270	32.163
α (DYNA/GLO)	1	3	5.829	9.470	13.924	19.182	25.247	32.120

times. These examples appear in Figs. 4 and 5, where it can be seen that the stresses never exceed the value of $2p$, a fact that qualifies them to be optimal designs. One can also observe that the time of onset of the maximum stress in each layer increases with the design parameter α. This means that if a design that delays the time of occurrence of the maximum stress is sought, then the value of the optimal design parameter α_n given in (27) should be maximized (within practical bounds). Because of such good agreement between the analytical results obtained previously and DYNA3D solutions, these findings support the use of purely computational means to establish optimal designs for transient problems of this nature.

Several other problems with different boundary conditions were then investigated using the DYNA3D/GLO software, including a two-layered free–free system with Heaviside loading, and two-layered free–free and free-fixed systems with rectangular pulse loading $p(t) = p(H(t) - H(t - t_0))$. Interestingly, none of these boundary-value problems provided optimal solutions with behavior similar to the optimality condition (27). However, for the free-fixed problem subjected to the rectangular pulse loading, as the value of t_0 increased and the loading approached that of a Heaviside loading, the optimal points (27) were gradually recovered.

4. Summary

In this paper, we investigated stress wave propagation in a two-layered elastic strip subjected to free-fixed boundary conditions under a Heaviside stress loading. Explicit expressions for the stress in each layer were obtained, and the countably infinite set of optimal designs which provide the smallest stress amplitude was identified. A time delay benefit when using optimal designs other than the homogeneous, was discovered. The analytical expressions for the stress in each layer compared well with the stress history profile predicted when using computational finite element methods. In addition, the optimality condition (27) provided means to validate a method based upon DYNA3D/GLO optimization software.

The validation of such computational optimization algorithms against known exact optimal solutions enables us to consider a broader class of impact and penetration problems using more realistic constitutive behaviors applicable to multilayered armor.

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